

Ex1 Given $x(t) = e^{-\alpha t} u(t)$, $\alpha > 0$.

(1) show the Parseval's relation.

(2) Find Energy of the signal at DC

(3) Find Energy of the signal at frequency

$$\omega = \alpha.$$

(4) Find Energy of the signal in interval $\omega_1 \leq \omega \leq \omega_2$.
where $\omega_2 = \frac{\pi}{2}$, $\omega_1 = \frac{\pi}{4}$.

Sol

$$E_x = \int_{t=-\infty}^{\infty} |x(t)|^2 dt = \int_0^{\infty} e^{-2\alpha t} dt = \frac{1}{2\alpha}$$

$$X(\omega) = \int_{t=0}^{\infty} e^{-\alpha t} e^{-j\omega t} dt = \frac{1}{\alpha + j\omega}$$

and

$$|X(\omega)|^2 = X(\omega) X(-\omega) = \left(\frac{1}{\alpha + j\omega} \right) \left(\frac{1}{\alpha - j\omega} \right) = \frac{1}{\alpha^2 + \omega^2}$$

Part 1

$$E_x = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} |X(\omega)|^2 d\omega = \left(\frac{1}{2\pi} \right) \left(\frac{1}{\alpha} \right) \left[\tan^{-1} \left(\frac{\omega}{\alpha} \right) \right]_{\omega=-\infty}^{\infty}$$

$$E_x = \left(\frac{1}{2\pi} \right) \left(\frac{1}{\alpha} \right) \left[\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right] = \frac{1}{2\alpha} = E_x$$

Part 2

$$|X(\omega)|^2 = \frac{1}{\alpha^2 + \omega^2} \Big|_{\omega=0} = \frac{1}{\alpha^2}$$

Part 3

$$|X(\omega)|^2 = \frac{1}{\alpha^2 + \omega^2} \Big|_{\omega=\alpha} = \frac{1}{2\alpha^2}$$

Part 4 of Ex 1:

$$E_x^2 = \frac{1}{2\pi} \int_{\omega=\frac{\pi}{4}}^{\frac{\pi}{2}} |X(\omega)|^2 d\omega = \left(\frac{1}{2\pi\alpha} \right) \left[\tan^{-1} \left(\frac{\omega}{\alpha} \right) \right]_{\omega=\frac{\pi}{4}}^{\omega=\frac{\pi}{2}}$$

$$= \left(\frac{1}{2\pi\alpha} \right) \left\{ \tan^{-1} \left(\frac{\pi}{2\alpha} \right) - \tan^{-1} \left(\frac{\pi}{4\alpha} \right) \right\}.$$

(9) Convolution:

If $x(t) \longleftrightarrow X(\omega)$ and $h(t) \longleftrightarrow H(\omega)$
then

$$y(t) = x(t) * h(t) \longleftrightarrow Y(\omega) = X(\omega) H(\omega)$$

Also: using FT Duality Property:

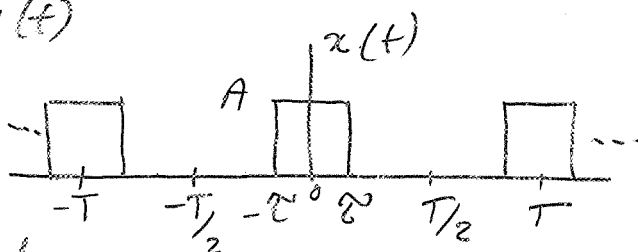
$$g(t) = x(t) \cdot h(t) \longleftrightarrow G(\omega) = \frac{1}{2\pi} [X(\omega) * H(\omega)]$$

Ex 1



where

$$h(t) = e^{-at} u(t) \text{ for } a > 0$$



- (1) Find spectrum of the output
- (2) Find average power of the 3rd harmonic of output when $T = 4\tau$.

Solution

Part 1 $h(t) = e^{-at} u(t) \leftrightarrow H(\omega) = \frac{1}{a + j\omega}$

The input $x(t)$ is periodic. Its Fourier Series coefficients are, T is fundamental period.

$$C_n = \frac{1}{T} \int_0^T x(t) e^{-j \frac{2\pi n}{T} t} dt$$

$$C_n = \frac{A}{T} \int_{-\tau}^{\tau} e^{-j \frac{2\pi n}{T} t} dt$$

$$C_n = \left(\frac{2A\tau}{T} \right) \frac{\sin\left(\frac{2\pi n}{T} \tau\right)}{\frac{2\pi n}{T} \tau} = (2A\tau F_0) \text{Sinc}(n\omega_0 \tau)$$

where $F_0 = \frac{1}{T}$ $\frac{2\pi}{T} = 2\pi F_0 = \omega_0$

We can write FT of $x(t)$ as

$$X(\omega) = \sum_{n=-\infty}^{\infty} C_n \delta(\omega - n\omega_0)$$

Since $y(t) = x(t) * h(t)$ then

$$Y(\omega) = X(\omega) \cdot H(\omega) = \sum_{n=-\infty}^{\infty} C_n \delta(\omega - n\omega_0) \left(\frac{1}{a + j\omega} \right)$$

$$Y(\omega) = \sum_{n=-\infty}^{\infty} C_n \left(\frac{1}{a + jn\omega_0} \right) \delta(\omega - n\omega_0)$$

or

$$Y(n\omega_0) = d_n = \frac{C_n}{a + jn\omega_0} = \left(\frac{2A\tau F_0}{a + jn\omega_0} \right) \text{Sinc}(n\omega_0 \tau)$$

for $-\infty < n < \infty$.

Part 2 Solution:

$$\omega_0 = \frac{2\pi}{T} = 2\pi F_0, \quad T = 4\mu\text{s}, \quad \text{Then } \omega_0 = \frac{\pi}{2\mu\text{s}}$$

$$Y(3\omega_0) = d_3 = \frac{C_3}{a + j3\omega_0} = \frac{C_3}{a + j\left(\frac{3\pi}{2\mu\text{s}}\right)}$$

$$Y(3\omega_0) = d_3 = \left(\frac{A}{3\pi}\right) \left(\frac{-j}{a + j3\omega_0}\right)$$

Average power of 3rd harmonic is

$$P_{3rd} = |d_3|^2 + |d_{-3}|^2 = 2|d_3|^2$$

$$= 2\left(\frac{A^2}{9\pi^2}\right) \left(\frac{1}{|a + j3\omega_0|^2}\right) = \left(\frac{2A^2}{9\pi^2}\right) \left(\frac{1}{a^2 + 9\omega_0^2}\right)$$

□

(10) Modulation

If $x(t) \longleftrightarrow X(\omega)$ and $m(t) \longleftrightarrow M(\omega)$

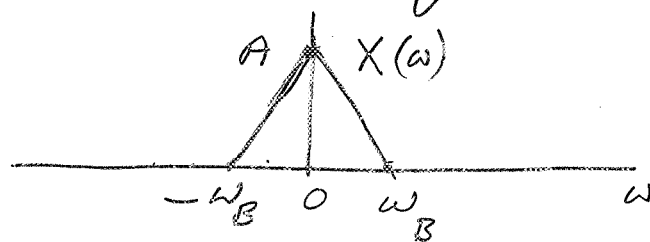
then $v(t) = x(t)m(t) \longleftrightarrow V(\omega) = \frac{1}{2\pi} \{ X(\omega) * M(\omega) \}$

$$V(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\alpha) M(\omega - \alpha) d\alpha$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega - \alpha) M(\alpha) d\alpha$$

Note See properties of Fourier Transform listed in Table 4.2.

Ex Let $x(t)$ be a band-limited signal whose spectrum is



then spectrum of

$$y(t) = \cos(\omega_0 t) x(t) \quad : \text{ i.e. real modulation}$$

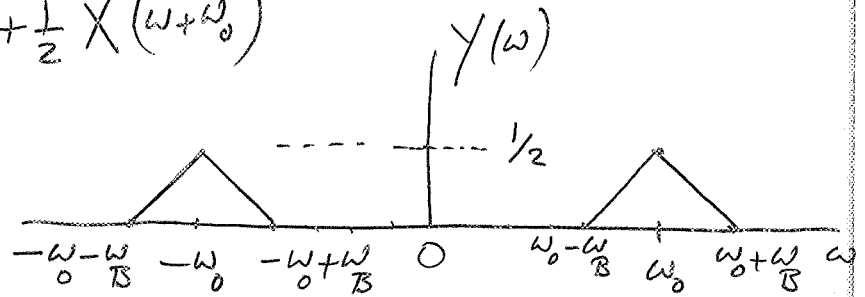
with $v(t) = \cos(\omega_0 t)$ is

$$Y(\omega) = \frac{1}{2\pi} V(\omega) * X(\omega).$$

$$\text{But } V(\omega) = \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)$$

Thus

$$Y(\omega) = \frac{1}{2} X(\omega - \omega_0) + \frac{1}{2} X(\omega + \omega_0)$$

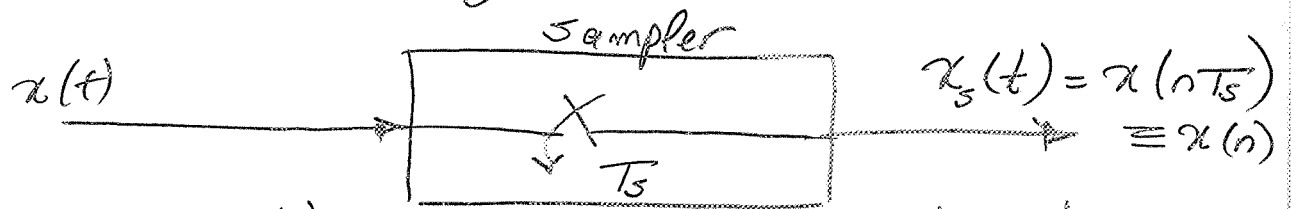


(II) Differentiation in Frequency:

If $x(t) \longleftrightarrow X(\omega)$

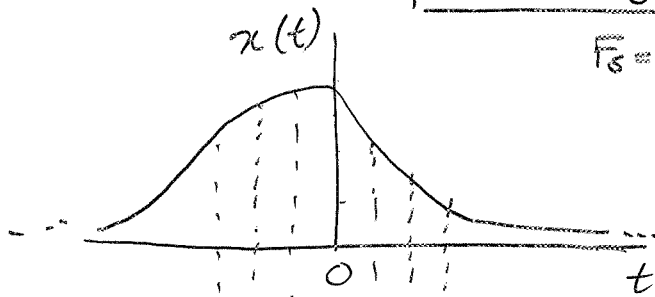
then $\omega(t) = (-jt)^n x(t) \longleftrightarrow Y(\omega) = \frac{d^n X(\omega)}{d\omega^n}$

Sampling Theorem

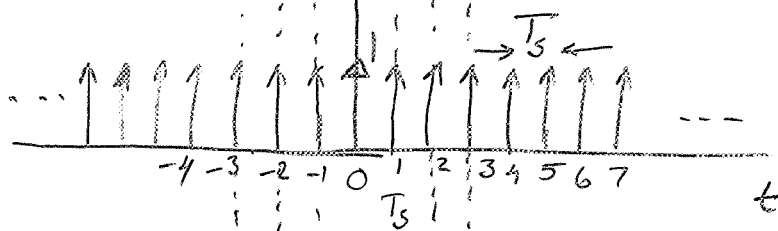


$F_s = \frac{1}{T_s}$ is sampling rate in Hz.

$x(t)$ is Band-limited signal

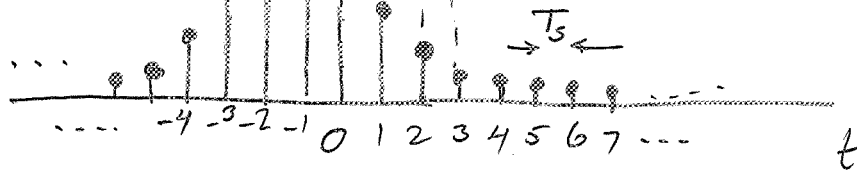


$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$



$$F_s = \frac{1}{T_s}$$

$$x_s(t) = x(t) \cdot p(t) = \sum_{n=-\infty}^{\infty} x(t) \delta(t - nT_s) = x(nT_s) \equiv x(n)$$



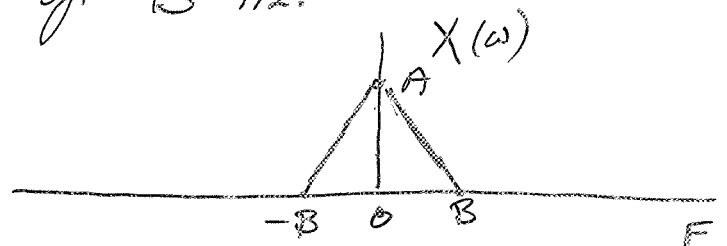
Note

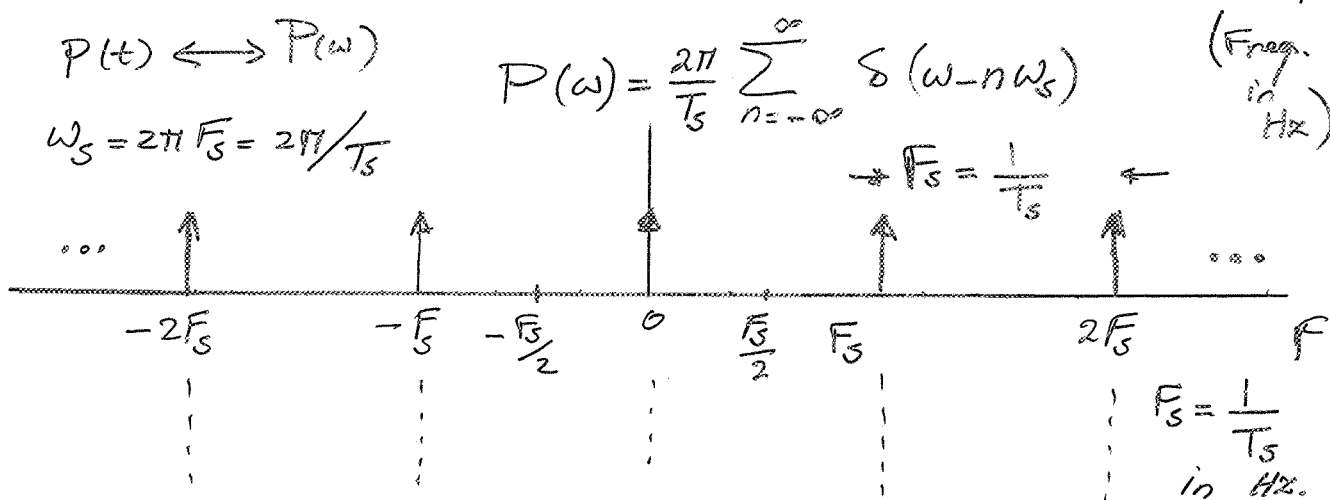
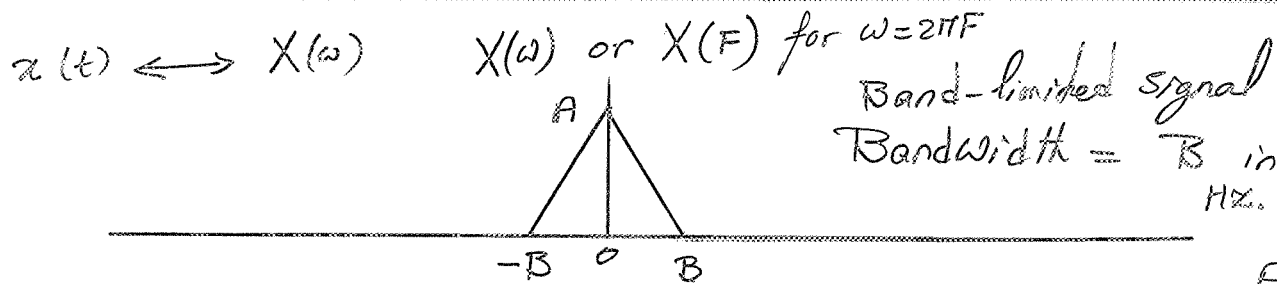
We assume that $x(t)$ is Band-limited with Bandwidth (BW) of B Hz.

$$x(t) \longleftrightarrow X(\omega)$$

Note

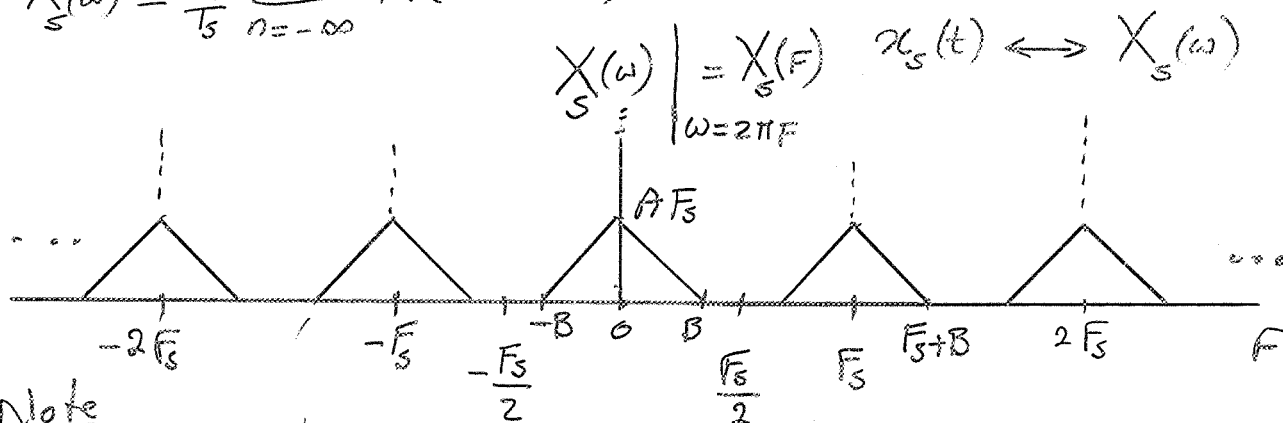
$F_s \geq 2B$ is called Nyquist rate.





$$X_s(\omega) = \frac{1}{2\pi} [P(\omega) * X(\omega)] = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega) * \delta(\omega - n\omega_s)$$

$$X_s(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$



Note

$$f = -2 \quad f = -1 \quad f = -\frac{1}{2} \quad 0 \quad f = \frac{1}{2} \quad f = 1 \quad f = \frac{F}{F_s}$$

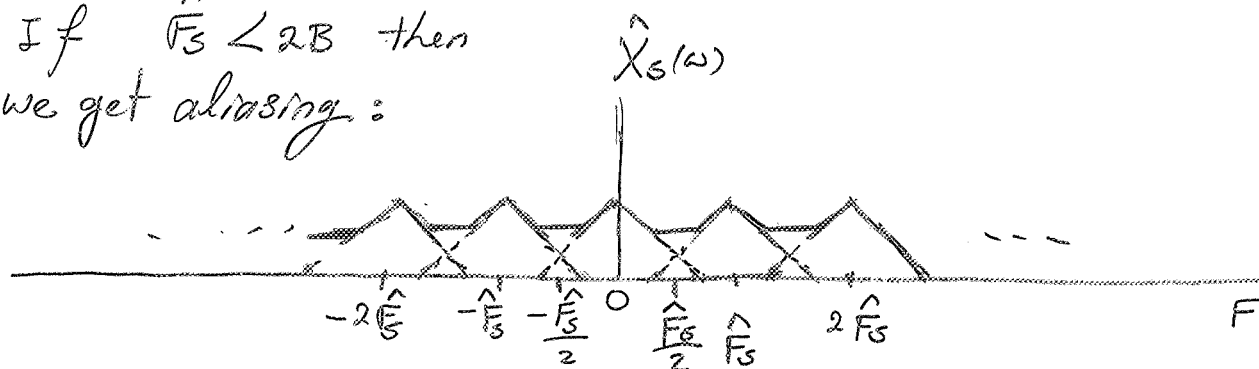
$$\Omega = -4\pi \quad \Omega = -2\pi \quad \Omega = -\pi \quad 0 \quad \Omega = \pi \quad \Omega = 2\pi \quad \Omega = 2\pi f$$

That is: $\Omega = 2\pi f = 2\pi \frac{F}{F_s} = 2\pi F T_s = \omega T_s = \frac{\omega}{F_s}$

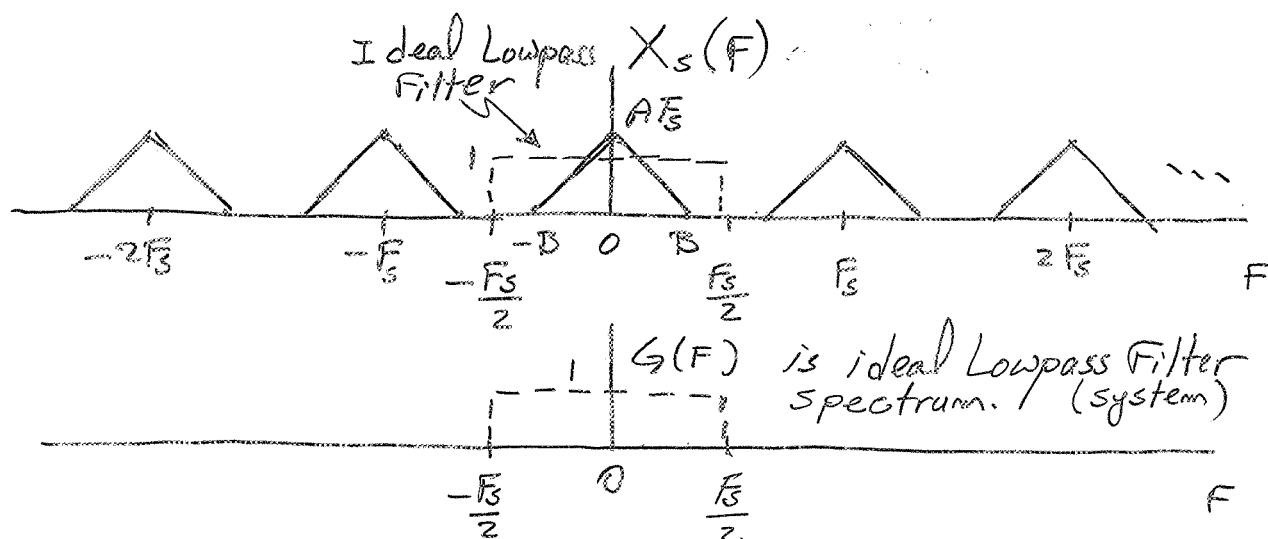
where $\omega = 2\pi F$ for continuous signal spectrum.
and Ω is for Discrete-Time signal spectrum.

Note For $\frac{F_s}{2} \geq B$, or $F_s \geq 2B$ called NYQUIST Rate
we get no aliasing distortion in Frequency domain for $X_s(\omega)$.

If $\hat{F}_s < 2B$ then
we get aliasing:



"Recovery the Continuous-time signal from"
the sampled (Discrete-time) signal



$$G(F) = \begin{cases} 1 & \text{for } -\frac{F_s}{2} \leq F \leq \frac{F_s}{2} \\ 0 & \text{elsewhere} \end{cases} \quad \text{or } G(\omega) = \begin{cases} 1 & |\omega| \leq \pi F_s \\ 0 & \text{elsewhere} \end{cases}$$

$G(F)$ is spectrum of an ideal Continuous-time
Lowpass filter.

Where $g(t) \xrightarrow{FT} G(F)$ (or $G(\omega) \big|_{\omega=2\pi F}$)

Thus

$$g(t) = \int_{F=-\infty}^{\infty} G(F) e^{j2\pi Ft} dF$$

$$= \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} G(\omega) e^{j\omega t} d\omega$$

Thus

$$g(t) = \int_{F=-\frac{F_s}{2}}^{\frac{F_s}{2}} G(F) e^{j2\pi Ft} dF = \int_{F=-\frac{F_s}{2}}^{\frac{F_s}{2}} e^{j2\pi Ft} dF$$

$$g(t) = \frac{1}{j2\pi t} \left\{ e^{j2\pi \frac{F_s}{2} t} - e^{-j2\pi \frac{F_s}{2} t} \right\}$$

$$g(t) = \frac{1}{j2\pi t} \left\{ 2j \sin(\pi F_s t) \right\}$$

$$g(t) = \frac{1}{\pi t} \sin(\pi F_s t) = F_s \operatorname{Sinc}(\pi F_s t) \quad \text{for } -\infty < t < \infty.$$

Recall That $F_s = \frac{1}{T_s}$ is sampling frequency.

Now

$$X(\omega) = \left(\frac{1}{F_s}\right) X_s(\omega) \cdot G(\omega)$$

or for $\omega = 2\pi F$:

$$X(F) = \left(\frac{1}{F_s}\right) X_s(F) \cdot G(F) \quad , \quad \frac{1}{F_s} = T_s$$

Thus in time-domain:

$$x(t) = \left(\frac{1}{F_s}\right) [x_s(t) * g(t)] = T_s [x_s(t) * g(t)]$$

that is

$$x(t) = T_s \int_{-\infty}^{\infty} g(\alpha) x_s(t-\alpha) d\alpha$$

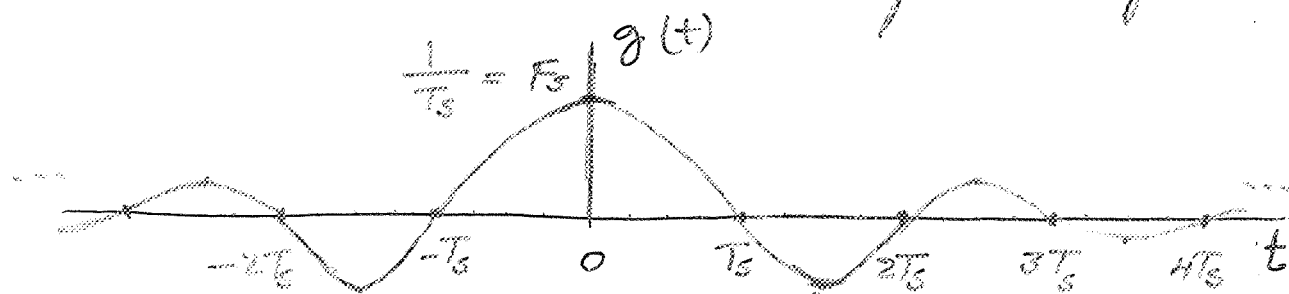
$$x(t) = T_s \int_{\alpha=-\infty}^{\infty} F_s \operatorname{Sinc}(\pi F_s \alpha) \sum_{n=-\infty}^{\infty} x(t-\alpha) \delta\left(t-\alpha-\frac{n}{F_s}\right) d\alpha$$

$$\text{or } x(t) = \sum_{n=-\infty}^{\infty} \int_{\alpha=-\infty}^{\infty} \text{Sinc}\left(\frac{\pi}{T_s} \alpha\right) x(t-\alpha) \delta(t-\alpha-nT_s) d\alpha$$

$\downarrow$
 $\alpha = t - nT_s$

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \text{Sinc}\left[\frac{\pi}{T_s}(t - nT_s)\right]$$

Note $g(t)$ is Impulse response of an Ideal Continuous-time Lowpass filter.
It is also called Interpolation function.

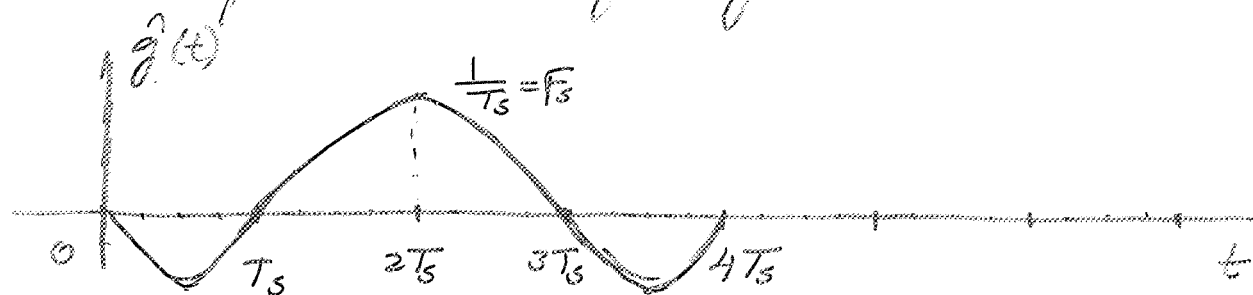


Note $g(t)$ is IIR and Non-causal, real, and even function of time.

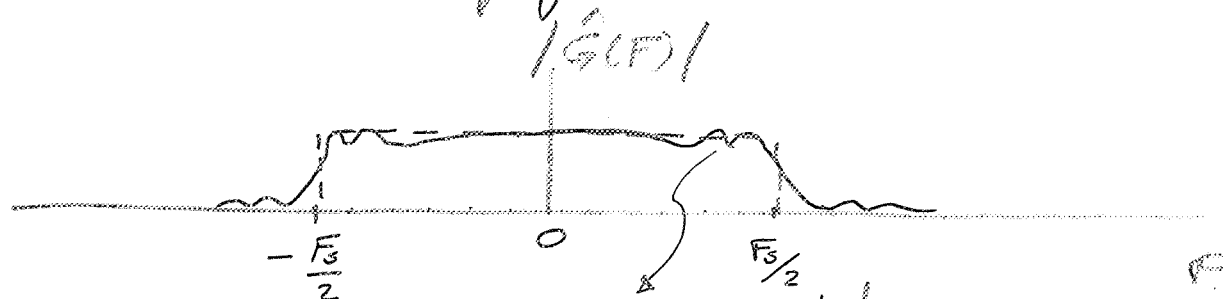
Thus we cannot realize the Ideal lowpass filter with impulse response $g(t)$ and Spectrum $G(F)$.

But we can approximate it by truncating $g(t)$ and shifting it to right to obtain a causal system (i.e. a causal impulse response).

For example we can get $\hat{g}(t)$ like



However the FT of $\hat{g}(t)$ is now $\hat{G}(F)$ that is:



Note

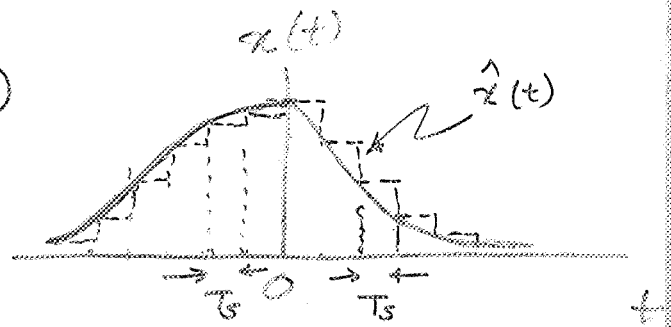
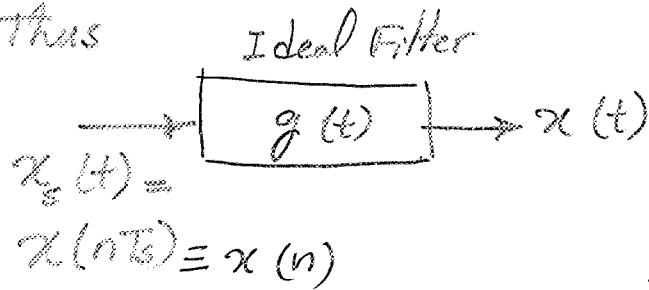
This oscillation is called
"Gibbs Phenomenon"
or Ringing

Caused by Truncating
the Sinc function $\hat{g}(t)$.

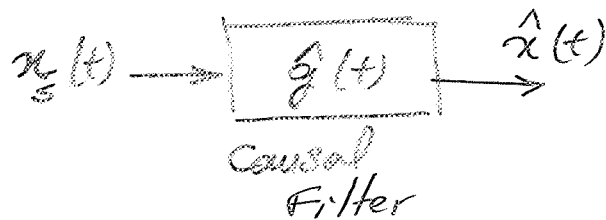
* The longer the length of $\hat{g}(t)$ the less Ringing, but with higher frequency of oscillation at the end sides of $\hat{G}(F)$.

* Also note that the shift to the right of $\hat{g}(t)$ creates a Linear phase shift for $\hat{G}(F)$.

Thus



But



$\hat{x}(t)$ looks like
stepping signal.

In practice, the following are often done:

- (1) T_s is taken small (i.e. high $F_s = \frac{1}{T_s}$ sampling rate) as much as possible to get $\hat{x}(t)$ closer to $x(t)$.
- (2) $\hat{g}(t)$ is usually impulse response of a simple RC filter (e.g. Zero-order hold filter).
- (3) Also another lowpass filter is used after $\hat{g}(t)$ to smooth out the corners of the $\hat{x}(t)$.
- (4) Also, $x(t)$ is (Lowpass) filtered before sampling to ensure $F_s \gg 2B$. This filter is called Anti-aliasing filter.